

Randomized methods for rank-deficient linear systems

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Abstract. We present a simple, accurate method for solving consistent, rank-deficient linear systems, with or without additional rank-completing constraints. Such problems arise in a variety of applications, such as the computation of the eigenvectors of a matrix corresponding to a known eigenvalue. The method is based on elementary linear algebra combined with the observation that if the matrix is rank- k deficient, then a random rank- k perturbation yields a nonsingular matrix with probability 1.

Key words. Rank-deficient systems, nullspace, null vectors, eigenvectors, randomized algorithms, integral equations.

1 Introduction

A variety of problems in numerical linear algebra involve the solution of rank-deficient linear systems. The most straightforward example is that of finding the eigenspace of a matrix $A \in \mathbb{C}^{n \times n}$ corresponding to a known eigenvalue λ . One then wishes to solve:

$$(A - \lambda I)x = 0.$$

If A itself is rank-deficient, of course, then setting $\lambda = 0$ corresponds to seeking its nullspace.

A second category of problems involves the solution of an inhomogeneous linear system

$$Ax = b, \tag{1}$$

where A is rank- k deficient but b is in the range of A . A third category consists of problems like (1), but for which a set of k additional constraints are known of the form:

$$C^*x = f, \tag{2}$$

where

$$\begin{pmatrix} A \\ C^* \end{pmatrix}$$

is full-rank. Here, $C \in \mathbb{C}^{n \times k}$, C^* denotes its adjoint, and $f \in \mathbb{C}^k$.

In this brief note, we describe a very simple framework for solving such problems, using *randomized* schemes. They are particularly useful when A is well-conditioned in a suitable $(n - k)$ -dimensional subspace. In terms of the singular value decomposition $A = U\Sigma V^*$, this corresponds to the case when σ_1/σ_{n-k} is of modest size and $\sigma_{n-k+1}, \dots, \sigma_n = 0$, where the $\{\sigma_i\}$ are the singular values of A . We do *not* address least squares problems, that is, we assume that the system (1), with or without (2), is consistent.

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Definition 1. We will denote by $\mathcal{N}(A)$ the nullspace of A and by $\mathcal{R}(A)$ its range.

There is a substantial literature on this subject, which we do not seek to review here. We refer the reader to the texts [8, 12] and the papers [1, 2, 3, 4, 5, 7, 11, 13, 14, 19]. Of particular note are [16, 17], which demonstrate the power of randomized schemes using methods closely related to the ones described below. It is also worth noting that, in recent years, the use of randomization together with numerical rank-based ideas has proven to be a powerful combination for a variety of problems in linear algebra (see, for example, [10, 15, 18]).

The basic idea in the present work is remarkably simple and illustrated by the following example. Suppose we are given a rank-1 deficient matrix A and that we carry out the following procedure:

1. Choose a random vector $x \in \mathbb{C}^n$ and compute $b = Ax$.
2. Choose random vectors $p, q \in \mathbb{C}^n$ and solve

$$(A + pq^*)y = b. \quad (3)$$

3. Then the difference $x - y$ is in the nullspace of A .

In order for $A + pq^*$ to be invertible, we must have that $p \notin \mathcal{R}(A)$ and $q \notin \mathcal{R}(A^*)$. Since p and q are random, this must occur with probability 1. It follows then that $A(x - y) = b - (b - pq^*y) = p(q^*y)$. Since $A(x - y)$ must be in $\mathcal{R}(A)$ and p is not, both sides vanish, implying that $x - y$ is a null-vector of A , and q^*y must be zero.

Another perspective, which may be more natural to some readers, is to consider the affine space $\{x' + \mathcal{N}(A)\}$, consisting of solutions to $Az = b$, where, x' is the solution of minimal norm. The difference of any two vectors in the affine space clearly lies in the nullspace of A . If $A + pq^*$ is nonsingular, y is the unique vector in the affine space orthogonal to q , implying that $x - y \in \mathcal{N}(A)$.

The remainder of this note is intended to make this procedure rigorous and to explore its extensions to related problems such as solving (1), (2).

2 Mathematical preliminaries

Much of our analysis depends on estimating the condition number of a rank- k deficient complex $n \times n$ matrix A to which is added a rank- k random perturbation. For $P, Q \in \mathbb{C}^{n \times k}$, we let

$$\begin{aligned} P &= P_R + P_{N^*}, & \mathcal{R}(P_R) &\subset \mathcal{R}(A), \mathcal{R}(P_{N^*}) \subset \mathcal{N}(A^*), \\ Q &= Q_{R^*} + Q_N, & \mathcal{R}(Q_{R^*}) &\subset \mathcal{R}(A^*), \mathcal{R}(Q_N) \subset \mathcal{N}(A), \end{aligned} \quad (4)$$

and

$$\begin{aligned} \rho &:= \|P_R\| = \sigma_{\max}(P_R), & \eta &:= \sigma_{\min}(P_{N^*}), \\ \xi &:= \|Q_{R^*}\| = \sigma_{\max}(Q_{R^*}), & \nu &:= \sigma_{\min}(Q_N), \end{aligned} \quad (5)$$

where all norms $\|\cdot\| = \|\cdot\|_2$.

Theorem 2. Let $b = Ax$ and let y be an approximate solution to

$$(A + PQ^*)y = b \quad (6)$$

in that it satisfies

$$\|b - (A + PQ^*)y\| \leq \delta. \quad (7)$$

Then

$$\|A(x - y)\| \leq \delta \left(1 + \frac{\|P\|}{\sigma_{\min}(P_{N^*})} \right). \quad (8)$$

Proof. It follows from (7) and the triangle inequality that

$$\|A(x - y)\| \leq \delta + \|P\| \|Q^*y\|. \quad (9)$$

Moreover,

$$b - Ay - P(Q^*y) = \delta f \quad (10)$$

for some vector $f \in \mathbb{C}^n$ with $\|f\| \leq 1$. Now let U be a matrix whose columns form an orthonormal basis for $\mathcal{N}(A^*)$. Multiplying on the left by U^* we have

$$-(U^*P)(Q^*y) = \delta(U^*f), \quad (11)$$

$$\|Q^*y\| \leq \frac{\delta}{\sigma_{\min}(P_{N^*})}, \quad (12)$$

where the last inequality follows from the fact that

$$\delta \geq \inf_{\|z\|=1, z \in \mathbb{C}^k} \|U^*Pz\| \|Q^*y\| = \inf_{\|z\|=1, z \in \mathbb{C}^k} \|UU^*Pz\| \|Q^*y\| = \sigma_{\min}(P_{N^*}) \|Q^*y\|, \quad (13)$$

which yields the desired result when combined with (9). \square

The obtained bound (8) indicates that $x - y$ is an approximate null-vector of matrix A , therefore, y is also an approximate solution to $Ay = b$ for a given consistent right-hand side $b \in \mathcal{R}(A)$.

Theorem 3. Let $A \in \mathbb{C}^{n \times n}$ have a k -dimensional nullspace and let $P, Q \in \mathbb{C}^{n \times k}$. Then

$$\|(A + PQ^*)^{-1}\| \leq \frac{1}{\sigma_{n-k}} \sqrt{1 + \left(\frac{\rho}{\eta}\right)^2 + \left(\frac{\xi}{\nu}\right)^2 + \left(\frac{\sigma_{n-k} + \rho\xi}{\eta\nu}\right)^2}, \quad (14)$$

where ρ, η, ξ, ν are defined in (5).

Proof. Let $A = U\Sigma V^*$ be the singular value decomposition of A . Let C and D be such that $P = UC$ and $Q = VD$. Let $C^T = [C_R^T \ C_{N^*}^T]$ where $C_R \in \mathbb{C}^{(n-k) \times k}$ and $C_{N^*} \in \mathbb{C}^{k \times k}$. The entries in the columns of C_R are coefficients of the corresponding columns of P in an orthonormal basis of the range of A . Thus $\|C_R\| = \rho$, and similarly $\|C_{N^*}^{-1}\| = 1/\eta$. Let $D^T = [D_{R^*}^T \ D_N^T]$ where $D_{R^*} \in \mathbb{C}^{(n-k) \times k}$ and $D_N \in \mathbb{C}^{k \times k}$. By similar reasoning, we have that $\|D_{R^*}\| = \xi$ and $\|D_N^{-1}\| = 1/\nu$

$$\|(A + PQ^*)^{-1}\| = \|(\Sigma + CD^*)^{-1}\| \quad (15)$$

and

$$(\Sigma + CD^*)^{-1} = \begin{pmatrix} \Sigma' + C_R D_{R^*}^* & C_R D_N^* \\ C_{N^*} D_{R^*}^* & C_{N^*} D_N^* \end{pmatrix}^{-1} \quad (16)$$

$$= \begin{pmatrix} \Sigma'^{-1} & -\Sigma'^{-1} C_R (C_{N^*})^{-1} \\ -(D_N^*)^{-1} D_{R^*}^* \Sigma'^{-1} & (D_N^*)^{-1} (I_k + D_{R^*}^* \Sigma'^{-1} C_R) (C_{N^*})^{-1} \end{pmatrix}, \quad (17)$$

where $\Sigma' \in \mathbb{C}^{(n-k) \times (n-k)}$ is the upper $(n-k) \times (n-k)$ sub-matrix of Σ and $I_k \in \mathbb{C}^{k \times k}$ is the identity matrix. This gives

$$\|(\Sigma + CD^*)^{-1}\| \leq \sqrt{\frac{1}{\sigma_{n-k}^2} + \left(\frac{\rho}{\sigma_{n-k}\eta}\right)^2 + \left(\frac{\xi}{\sigma_{n-k}\nu}\right)^2 + \left(\frac{1 + \rho\xi/\sigma_{n-k}}{\eta\nu}\right)^2} \quad (18)$$

$$= \frac{1}{\sigma_{n-k}} \sqrt{1 + \left(\frac{\rho}{\eta}\right)^2 + \left(\frac{\xi}{\nu}\right)^2 + \left(\frac{\sigma_{n-k} + \rho\xi}{\eta\nu}\right)^2}. \quad (19)$$

\square

It follows from this result that one can bound the conditioning of the perturbed matrix.

Theorem 4. *Let $A \in \mathbb{C}^{n \times n}$ have a k -dimensional nullspace and let $P, Q \in \mathbb{C}^{n \times k}$. Then*

$$\kappa(A + PQ^*) \leq \frac{\sigma_1 + \|P\| \|Q\|}{\sigma_{n-k}} \sqrt{1 + \left(\frac{\rho}{\eta}\right)^2 + \left(\frac{\xi}{\nu}\right)^2 + \left(\frac{\sigma_{n-k} + \rho\xi}{\eta\nu}\right)^2}, \quad (20)$$

where ρ, η, ξ, ν are defined in (5).

The preceding theorems indicate that, in the absence of additional information, it is reasonable to pick random vectors of approximately unit norm and to scale the perturbation term PQ^* by the norm of A .

3 Solving consistent, rank-deficient linear systems

Let us first consider the solution of the rank- k deficient linear system $Ax = b$ in the special case where $\mathcal{N}(A)$ and $\mathcal{N}(A^*)$ are spanned by the columns of known matrices N and V , respectively. Suppose, now, that we solve the linear system

$$(A + VN^*)x = b. \quad (21)$$

Consistency here requires that $V^*Ax = V^*b = 0$, so that $(V^*V)(N^*x) = 0$, from which $N^*x = 0$. Thus, x is the particular solution to $Ax = b$ that is orthogonal to the nullspace of A . From Theorem 4, the condition number of $A + VN^*$ is given by

$$\kappa(A + VN^*) \leq \frac{\sigma_1 + \|V\| \|N\|}{\sigma_{n-k}} \sqrt{1 + \left(\frac{\sigma_{n-k}}{\sigma_{\min}(V)\sigma_{\min}(N)}\right)^2}. \quad (22)$$

Remark 5. Note that this procedure allows us to obtain the minimum norm solution to the underdetermined linear system without recourse to the SVD or other dense matrix methods. Any method for solving (21) can be used. Assuming that $(A + VN^*)$ is reasonably well conditioned and that A can be applied efficiently, Krylov space methods such as GMRES are extremely effective.

Suppose now that we have no prior information about the nullspaces of A and/or A^* . We may then substitute random matrices P and Q for V and/or N and follow the same procedure. With probability 1, $(A + PQ^*)$ will be invertible and we will obtain the particular solution to $Ax = b$ that is orthogonal to Q . This simply requires that the projections of P onto $\mathcal{N}(A^*)$ and of Q onto $\mathcal{N}(A)$, denoted by P_{N^*} and Q_N respectively, must be full-rank (see (4)).

3.1 Consistent, rectangular linear systems

We next consider the case where we wish to solve the system (1) together with (2). Note that, for consistency, we must still have that $V^*A = V^*b = 0$, where the columns of V span $\mathcal{N}(A^*)$. Note also that the system

$$\begin{pmatrix} A \\ C^* \end{pmatrix} x = \begin{pmatrix} b \\ f \end{pmatrix} \quad (23)$$

is full-rank if and only if any vector in $\mathcal{N}(A)$ has a nontrivial projection onto the columns of C . There is no need, however, to solve a rectangular system of equations (23). One need only solve the $n \times n$ linear system

$$(A + VC^*)x = b + Vf.$$

From Theorem 4, the condition number of $A + VC^*$ is given by

$$\kappa(A + VC^*) \leq \frac{\sigma_1 + \|V\| \|C\|}{\sigma_{n-k}} \sqrt{1 + \left(\frac{\xi}{\sigma_{\min}(C_N)}\right)^2 + \left(\frac{\sigma_{n-k}}{\sigma_{\min}(V)\sigma_{\min}(C_N)}\right)^2}, \quad (24)$$

where ξ is the norm of C_{R^*} . In some applications, the data may be known to be consistent (b is in the range of A), but V may not be known. Then, one can proceed, as above, by solving

$$(A + PC^*)x = b + Pf,$$

where P is a random $n \times k$ matrix. From Theorem 4, the condition number of $A + PC^*$ is given by

$$\kappa(A + PC^*) \leq \frac{\sigma_1 + \|P\| \|C\|}{\sigma_{n-k}} \sqrt{1 + \left(\frac{\rho}{\sigma_{\min}(P_{N^*})}\right)^2 + \left(\frac{\xi}{\sigma_{\min}(C_N)}\right)^2 + \left(\frac{\sigma_{n-k} + \rho\xi}{\sigma_{\min}(P_{N^*})\sigma_{\min}(C_N)}\right)^2} \quad (25)$$

where ρ and ξ are the norms of P_R and C_{R^*} , respectively.

4 Computing the nullspace

Let us return now to the question of finding a basis for the nullspace of a rank- k deficient matrix $A \in \mathbb{C}^{n \times n}$. As in the introduction, we begin by describing the procedure.

1. Choose k random vectors $\{x_i, i = 1, \dots, k\} \in \mathbb{C}^n$ and compute $b_i = Ax_i$.
2. Choose random matrices $P, Q \in \mathbb{C}^{n \times k}$ and solve

$$(A + PQ^*)y_i = b_i. \quad (26)$$

Then, $A(x_i - y_i) = b_i - (b_i - PQ^*y_i) = P(Q^*y_i)$. Since $A(x_i - y_i) \in \mathcal{R}(A)$, and assuming $P(Q^*y_i) \notin \mathcal{R}(A)$, it follows that both sides must equal zero and that each vector $z_i = x_i - y_i$ is a null vector. Since the construction is random, the probability that the $\{z_i\}$ are linearly independent is 1. The result $P(Q^*y_i) \notin \mathcal{R}(A)$ follows from the fact that P is random and that the projection of each column of P onto $\mathcal{N}(A^*)$ will be linearly independent with probability 1. Theorem 4 tells us how to estimate the condition number of (26). Finally, the accuracy of the nullspace vectors $\{z_i\}$ can be further improved by an iterative refinement $\tilde{z}_i = z_i - \tilde{y}_i$, where the correction vectors \tilde{y}_i solve (26)

$$(A + PQ^*)\tilde{y}_i = \tilde{b}_i, \quad (27)$$

with the updated right-hand sides $\tilde{b}_i = Az_i$.

4.1 Determining the dimension of the nullspace

When the dimension of the nullspace is unknown, the algorithm above can also be used as a *rank-revealing* scheme. For this, suppose that the actual rank-deficiency is known to be k_A and that we carry out the above procedure with $k > k_A$. The argument that $P(Q^*y_i) \notin \mathcal{R}(A)$ will fail, since the projection of each of the columns of P onto $\mathcal{N}(A^*)$ must be linearly dependent. As a result, $x_i - y_i$ will fail to be a null-vector (which will be obvious from the explicit computation of $A(x_i - y_i)$). The estimated rank k can then be systematically reduced to determine k_A . If k_A is large, bisection can be used to accelerate this estimate.

4.2 Stabilization

Since the condition number of the randomly perturbed matrix is controlled only in a probabilistic sense, if high precision is required one can use a variant of iterative refinement to improve the solution. That is, one can first compute q_1, \dots, q_k as approximate null-vectors of A and p_1, \dots, p_k as approximate null-vectors of A^* . With these at hand, one can repeat the calculation with P and Q whose columns are $\{p_1, \dots, p_k\}$ and $\{q_1, \dots, q_k\}$, respectively. The parameters ρ/η and ξ/ν in Theorem 4 will be much less than 1 and the condition number of a second iteration will be approximately

$$\kappa(A + PQ^*) \approx \frac{\sigma_1 + \|P\| \|Q\|}{\sigma_{n-k}} \sqrt{1 + \left(\frac{\sigma_{n-k}}{\sigma_{\min}(P_{N^*})\sigma_{\min}(Q_N)}\right)^2}. \quad (28)$$

5 Numerical experiments

In this section, we describe the results of several numerical tests of the algorithms discussed above. All computations were performed in IEEE double-precision arithmetic using MATLAB version R2012a ¹.

We use a pseudorandom number generator to create $n \times 1$ vectors $\phi_1, \phi_2, \dots, \phi_{n-k}$ and $\psi_1, \psi_2, \dots, \psi_{n-k}$, with entries that are independent and identically distributed Gaussian random variables of zero mean and unit variance. We apply the Gram-Schmidt process with reorthogonalization to $\phi_1, \phi_2, \dots, \phi_{n-k}$ and $\psi_1, \psi_2, \dots, \psi_{n-k}$ to obtain orthonormal vectors u_1, u_2, \dots, u_{n-k} , and v_1, v_2, \dots, v_{n-k} , respectively. We define A to be the $n \times n$ matrix

$$A = \sum_{i=1}^{n-k} u_i \sigma_i v_i^*, \quad (29)$$

where $\sigma_i = 1/i$. The rank deficiency of A is clearly equal to k .

In Table 1, we compare the regular and stabilized versions of the new algorithm for finding the nullspace of a rank-deficient matrix A . The first and second columns contain the parameters n and k determining the size and the rank deficiency of problem, respectively. The third column contains the modified condition number σ_1/σ_{n-k} of the original matrix A , ignoring the zero singular values for more meaningful comparison between columns. The fourth column contains the true condition number σ_1/σ_n of a random rank- k perturbation $A + PQ^*$. Finally, the fifth and sixth columns contain the relative accuracy $\|AN\|/\|N\|$ in determining the nullspace vectors N for the randomized rank- k correction scheme before and after iterative refinement, respectively.

In Table 2, we compare the accuracy of the regular and stabilized versions of the randomized rank- k correction scheme for solving a rank-deficient linear system $Ax = b$ with a consistent right hand side b . The first and second columns contain the parameters n and k determining the size and the rank deficiency of problem, respectively. The third and fourth columns contain the modified condition number σ_1/σ_{n-k} of the original matrix A and the condition number σ_1/σ_n of a random rank- k perturbation $A + PQ^*$, respectively. The fifth column contains the condition number σ_1/σ_n of the rank- k perturbation $A + VN^*$, where V and N are the approximate null-vectors spanning the left and right nullspaces, respectively. Finally, the sixth and seventh columns contain the relative accuracy $\|Ax - b\|/\|b\|$ in determining the solution vector x for the regular and stabilized schemes, respectively.

It is clear from Table 2 that the condition number can be quite large for the non-stabilized version of the algorithm when the rank deficiency is high. This is due to the difficulty of finding high-dimensional random matrices P and Q that have large projections onto the corresponding nullspaces $\mathcal{N}(A^*)$ and $\mathcal{N}(A)$. In such cases, the algorithm will strongly benefit from the stabilization procedure.

6 Further examples

Our interest in the development of randomized methods was driven largely by issues in the regularization of integral equation methods in potential theory. For illustration, consider the Neumann problem for the Laplace equation in the interior of a simply-connected, smooth domain $\Omega \subset \mathbb{R}^2$ with boundary Γ .

$$\Delta u = 0 \text{ in } \Omega, \quad \frac{\partial u}{\partial n} = f \text{ on } \Gamma.$$

Classical potential theory [9] suggests seeking the solution as a single layer potential

$$u(x) = \frac{1}{2\pi} \int_{\Gamma} \log \|x - y\| \sigma(y) ds_y.$$

Using standard jump relations, this results in the integral equation

$$\sigma(x) + \frac{1}{\pi} \int_{\Gamma} \frac{\partial}{\partial n_x} \log \|x - y\| \sigma(y) ds_y = 2f(x), \quad (30)$$

¹Any mention of commercial products or reference to commercial organizations is for information only; it does not imply recommendation or endorsement by NIST.

n	k	$\text{cond}(A)$	$\text{cond}(A + PQ^*)$	E_2	$E_2(\text{refined})$
160	1	$1.590E + 02$	$2.025E + 03$	$1.368E - 16$	$8.106E - 17$
160	3	$1.570E + 02$	$4.258E + 04$	$2.180E - 15$	$2.727E - 16$
160	6	$1.540E + 02$	$1.144E + 04$	$2.706E - 14$	$6.382E - 16$
320	1	$3.190E + 02$	$5.259E + 03$	$9.072E - 17$	$3.556E - 17$
320	3	$3.170E + 02$	$9.340E + 03$	$1.983E - 16$	$6.029E - 17$
320	6	$3.140E + 02$	$3.374E + 04$	$7.461E - 16$	$2.471E - 16$
640	1	$6.390E + 02$	$3.968E + 04$	$1.934E - 16$	$2.099E - 16$
640	3	$6.370E + 02$	$1.332E + 06$	$3.879E - 15$	$5.817E - 16$
640	6	$6.340E + 02$	$3.899E + 06$	$5.924E - 13$	$5.781E - 16$
1280	1	$1.279E + 03$	$6.003E + 06$	$5.549E - 16$	$3.244E - 16$
1280	3	$1.277E + 03$	$4.998E + 04$	$1.023E - 14$	$6.990E - 17$
1280	6	$1.274E + 03$	$6.515E + 05$	$3.706E - 15$	$8.126E - 16$
160	75	$8.500E + 01$	$2.394E + 05$	$4.208E - 13$	$2.118E - 14$
160	80	$8.000E + 01$	$3.199E + 04$	$2.185E - 13$	$2.480E - 15$
320	155	$1.650E + 02$	$1.445E + 06$	$3.228E - 12$	$7.493E - 15$
320	160	$1.600E + 02$	$1.607E + 06$	$1.465E - 11$	$1.578E - 14$
640	315	$3.250E + 02$	$1.006E + 07$	$1.155E - 11$	$6.883E - 15$
640	320	$3.200E + 02$	$4.288E + 06$	$1.602E - 11$	$1.939E - 14$
1280	635	$6.450E + 02$	$3.551E + 08$	$2.714E - 10$	$4.323E - 14$
1280	640	$6.400E + 02$	$1.873E + 08$	$1.902E - 11$	$5.665E - 14$

Table 1: *Relative errors in determining the nullspace vectors for the randomized rank- k correction scheme before and after iterative refinement.*

n	k	$\text{cond}(A)$	$\text{cond}(A + PQ^*)$	E_2	$\text{cond}(A + UV^*)$	$E_2(\text{stab})$
160	1	$1.590E + 02$	$9.017E + 02$	$1.282E - 15$	$1.590E + 02$	$1.141E - 15$
160	3	$1.570E + 02$	$3.121E + 03$	$3.890E - 15$	$1.570E + 02$	$1.910E - 15$
160	6	$1.540E + 02$	$1.284E + 06$	$1.487E - 13$	$1.540E + 02$	$1.656E - 15$
320	1	$3.190E + 02$	$4.956E + 05$	$7.388E - 15$	$3.190E + 02$	$1.209E - 15$
320	3	$3.170E + 02$	$4.059E + 05$	$6.638E - 14$	$3.170E + 02$	$2.939E - 15$
320	6	$3.140E + 02$	$3.271E + 04$	$1.100E - 14$	$3.140E + 02$	$2.704E - 15$
640	1	$6.390E + 02$	$1.232E + 05$	$1.758E - 14$	$6.390E + 02$	$2.072E - 15$
640	3	$6.370E + 02$	$8.812E + 04$	$9.113E - 15$	$6.370E + 02$	$3.085E - 15$
640	6	$6.340E + 02$	$1.622E + 05$	$9.870E - 15$	$6.340E + 02$	$2.797E - 15$
1280	1	$1.279E + 03$	$8.325E + 04$	$4.545E - 15$	$1.279E + 03$	$3.483E - 15$
1280	3	$1.277E + 03$	$5.174E + 05$	$1.714E - 14$	$1.277E + 03$	$6.914E - 15$
1280	6	$1.274E + 03$	$7.675E + 05$	$3.905E - 14$	$1.274E + 03$	$4.661E - 15$
160	75	$8.500E + 01$	$7.057E + 04$	$3.854E - 13$	$8.500E + 01$	$4.157E - 15$
160	80	$8.000E + 01$	$2.357E + 04$	$9.249E - 14$	$8.000E + 01$	$3.975E - 15$
320	155	$1.650E + 02$	$1.732E + 05$	$1.886E - 13$	$1.650E + 02$	$1.208E - 14$
320	160	$1.600E + 02$	$9.449E + 05$	$6.109E - 12$	$1.600E + 02$	$8.945E - 15$
640	315	$3.250E + 02$	$5.510E + 07$	$8.537E - 11$	$3.250E + 02$	$2.612E - 14$
640	320	$3.200E + 02$	$2.623E + 07$	$1.591E - 11$	$3.200E + 02$	$1.884E - 14$
1280	635	$6.450E + 02$	$5.970E + 06$	$7.540E - 12$	$6.450E + 02$	$3.236E - 14$
1280	640	$6.400E + 02$	$1.134E + 07$	$1.162E - 11$	$6.400E + 02$	$7.476E - 14$

Table 2: *Relative errors for the regular and stabilized versions of the randomized rank- k correction scheme in determining the solution of the rank- k deficient linear system $Ax = b$ with the consistent right-hand side $b \in \mathcal{R}(A)$.*

which we write as

$$(I + K)\sigma = 2f.$$

It is well-known that (30) is solvable if and only if the right-hand side satisfies the compatibility condition: $\int_{\Gamma} f(y) ds_y = 0$. Using the L_2 inner product (for real-valued functions)

$$\langle f, g \rangle = \int_{\Gamma} f(y)g(y) ds_y,$$

we may write the compatibility condition as

$$\langle 1, f \rangle = 0,$$

where 1 denotes the function that is identically 1 on Γ . The function 1 is also in the nullspace of $I + K^*$, the adjoint of the integral operator in (30), which is clearly necessary for solvability. Following the procedure in section 3, we regularize the integral equation by solving

$$\sigma(x) + \frac{1}{\pi} \int_{\Gamma} \frac{\partial}{\partial n_x} \log \|x - y\| \sigma(y) ds_y + \int_{\Gamma} [r(x)1(y)] \sigma(y) dy = 2f(x), \quad (31)$$

or

$$(I + K)\sigma + r(x)\langle 1, \sigma \rangle = 2f,$$

where $r(x)$ is a random function defined on Γ . Taking the inner product of (31) with the function 1 yields

$$\langle 1, r \rangle \langle 1, \sigma \rangle = 0.$$

This is a well-known fact for the Neumann problem, and the obvious choice is simply $r(x) = 1$ so that (31) becomes:

$$\sigma(x) + \frac{1}{\pi} \int_{\Gamma} \left[\frac{\partial}{\partial n_x} \log \|x - y\| + 1 \right] \sigma(y) ds_y = 2f(x).$$

For an application of the preceding analysis in electromagnetic scattering, see [20]. In [6], a situation of the type discussed in section 3.1 arises. Without entering into details, it was shown that the “magnetic field integral equation” is rank- k deficient in the static limit in exterior multiply-connected domains of genus k . A set of k nontrivial constraints was derived from electromagnetic considerations, which were added to the system matrix as described above. Since we have illustrated the basic principle in the context of the nullspace problem, we omit further numerical calculations.

7 Conclusions

We have presented a simple set of tools for solving rank-deficient, but consistent, linear systems and demonstrated their utility with some numerical examples. Since the perturbed/augmented linear systems are reasonably well-conditioned with high probability, one can rely on Krylov subspace based iterative methods (e.g., conjugate gradient for self-adjoint problems or GMRES for non self-adjoint problems), avoiding the cost of dense linear algebraic methods, such as Gaussian elimination or the SVD itself. This is a particularly powerful approach when A is sparse or there is a fast algorithm for applying A to a vector. Finite rank-deficiency issues arise in the continuous setting as well, especially in integral equation methods, which we have touched on only briefly here.

We are currently working on the development of robust software for the nullspace problem that we expect will be competitive with standard approaches such as QR-based schemes [2], inverse iteration [5, 8] or Arnoldi methods [7].

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